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Advertising expenditures : a game of strategy.

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THESIS

ADVERTISING EXPENDITURES - A GAME OF STRATEGY

by

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October 1969

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Advertising Expenditures - A Game of Strategy

by

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ABSTRACT

This paper looks at the problem of optimally allocating advertising funds from a game theory point of view. Two basic models are presented and then expanded upon. These models are simply structured having originated from work done on the Colonel Blotto Game in the early 1950's. A prime objective of this paper is to briefly review representative examples of work previously done in this area and indicate the possible direction of future research. One of the more interesting extensions of the basic model is the development of a relation between the amount of money spent on advertising and profit.

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I. INTRODUCTION

It is a recognized fact in the business community that advertising is a powerful tool in the area of product marketing. Companies spend large sums of money annually in an effort to procure the patronage of various segments of the population. The advertising staff is a vital unit of most large companies; without such a staff it would be difficult to survive in the highly competitive environment of today's business world. A manager who is able to establish a successful program of advertising is well rewarded. However, should a manager be unable to develop and maintain an advertising program which is at least as effective as his competitors' he may find his business career cut short. But even considering the importance of advertising, the majority of executives who work in the field of advertising often rely on past experience and knowledge of the market as a basis for their recommendations and decisions affecting the allocation of advertising funds. Of course these executives back up their experience with extensive market research. A brief examination of past history shows that this method of deciding upon the quantity of money to be spent on advertising has many times been most successful. However it would seem possible that some purely analytic, and somewhat less subjective, techniques could be applied to the problem of optimally allocating advertising expenditures. One suggested approach is Game Theory.

It is the purpose of this paper to examine possible applications of certain game theory models to the problem of optimally allocating advertising funds. It is not feasible in this paper to cover all

possible applications of Game Theory to problems of advertising, rather
it is hoped that a broad overview of such applications can be presented
so as to act as a basis for future work in this area.

The specific results of the models presented in this paper are
probably more applicable to national advertising as done by large
corporations rather than that advertising done by small businesses on
a local level. However the structure of the models is probably ap-
plicable in all cases.

It is assumed that readers of this paper will have a background
in mathematics and hopefully be acquainted with Game Theory. As an
aid to those readers who are not familiar with Game Theory, the fol-
lowing section of this paper has been devoted to a summary of some
of the more important definitions and concepts associated with this
branch of mathematics. This summary should not be considered to be
complete, therefore the reader should be prepared to consult the
references listed in the bibliography.

II. NATURE OF GAME THEORY

The Theory of Games, perhaps more correctly called the Theory of Games of Strategy, may be described as a mathematical theory of decision making by participants in a competitive environment. In a typical problem to which this theory is applicable, each participant can bring some influence to bear upon the outcome of a certain event. No single competitor by himself, nor chance alone, can determine the outcome of the conflict situation completely. The Theory of Games of Strategy is then concerned with the problems of choosing an optimal course of action which takes into account the possible actions of the various participants, along with certain chance occurrences.

Some examples of games of strategy are poker, chess, bridge, and price competition between two sellers. It is to be remembered that games of strategy allow the players to make use of their ingenuity in order to influence the final outcome of the conflict situation established by the "game." Note that several of the examples of games of strategy given above involve the element of chance, for example poker (and most other card games), but none the less in each case the participants in these games are allowed, under the rules of the game, to make certain decisions which are completely independent of chance. Games which depend completely on chance and which do not allow the participants an opportunity to exercise any influence, such as roulette or dice, are not considered in the Theory of Games of Strategy. The mathematical theory of probability was developed for the study of games of the type just described - games of chance.

Although strategic situations have long been observed and recorded, the first attempt to abstract them into a mathematical theory of strategy was made by Émile Borel in 1921. The theory was firmly established in 1928 by Von Neumann when he proved the Minimax Theorem, the fundamental theorem of the Theory of Games of Strategy. This theorem is basic because it states that optimal solutions exist for all situations appropriate to Game Theory.

Before preceeding further it is necessary to present a few definitions and concepts for those unfamiliar with Game Theory.

First of all, a game of strategy is described by its set of rules. The rules specify what each participant, called a player, is allowed, or required, to do under all possible circumstances. Further, rules determine the amount of information, if any, each player receives. If the game involves the use of chance devices, or if chance occurrences are an integral part of the situation establishing the game, the rules specify how the chance events shall be interpreted. Finally, the rules define when the game ends, the amount each player pays or receives, and the objective of each player.

A game of strategy, from here on referred to simply as games, may be either finite or infinite. A game is finite if each player has a finite number of actions open to him at each play of the game. Similarly, a game is infinite if each player has an infinite number of actions available to him at each play of the game.

In addition to being classified as either finite or infinite, a game is classified according to the number of players participating in the game. Many games involve only two players, and

these are naturally called <u>Two-person Games</u>. Games in which more than two parties take part are customarily called <u>N-person Games</u>, N = 3, 4, 5,...... .

Throughout the above discussion it has been implied, but never stated, that certain units, such as dollars, are exchanged between the players of a game. If the loss of one player, or group of players, equals the winnings of the remaining player, or group of players, then the game is said to be <u>Zero-sum</u>. When it is said that the winnings and losses are equal it is meant that they are equal in the sense of utility. If the total sum of gains and losses is not zero then the game is said to be <u>Non-Zero-sum</u>.

To simplify the description of a game the concept of a strategy is now introduced. Each player develops, in advance of a play of the game, a plan for playing the game from beginning to end. This plan must be complete and cover all possible contingencies which may arise during a play of the game. The plan would make use of any information which may become available to the player in accordance with the rules of the game. Such a complete plan for a play of a game by a player is called a <u>strategy</u> for that player. Please note that a player's plan of action, his strategy, is complete and ready to use before the commencement of the game.

A strategy which guarantees a player the best he can expect regardless of what the other players do is called an <u>optimal strategy</u>.

The expression "a play of the game" has been used several times already in this paper. However, the exact nature of a play of the game may or may not be apparent. In the Theory of Games the choosing

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of a particular strategy by each player, along with the exchange of pay-offs which possibly result, is defined as a play of the game.

The value of the game is the expected payoff transferred between the players when each player employs his optimal strategy.

Usually the opposing players are placed into one of two catagories, either maximizing or minimizing. In an unfair game, a game in which the value is some number greater than zero, the maximizing player, or group of players, will realize a positive expectation. Therefore, the maximizing player will select an optimal strategy so as to max-imize his winnings. On the other hand in an unfair game the minimiz-ing player, or groups of players, will expect to lose the value of the game. Therefore, the minimizing player will choose an optimal strategy in order to minimize his losses. Of course if the value of the game is negative, then the minimizing player will have negative expected losses and the maximizing player will have negative expected winnings. In this case the maximizing player will continue to select strategies which will maximize his expected winnings. But since his expected winnings are negative, in this case, he is in effect mini-mizing his losses. Likewise the minimizing player will continue to choose strategies which will minimize his expected losses. But since his expected losses are negative he is actually maximizing his winnings. In either case, and also in the case of a fair game, one in which the value of the game is zero, all players select strategies which will maximize their individual utilities.

Finally, the solution of a game is an optimal strategy for each player and a real number which represents the value of the game.

III. BASIC ALLOCATION MODELS

One of the main objectives of the various companies comprising an
industry is to obtain as large a portion of the potential sales market
as possible. Obviously they will in general attempt to attain this
goal through the use of advertising. Since a given quantity, and
quality, of advertising costs some specific amount of money, the
companies will find their advertising programs constrained by the
amount of available funds. Because the amount of money available
for advertising is limited, it seems reasonable to assume that the
various competitors will attempt to tailor their advertising programs
so as to take the maximum advantage of one another's advertising
errors. These errors being mistakes in the allocation of advertising
expenditures.

A conflict situation has now been described in which the parti-
cipants are able to influence the outcome by selecting various al-
locations of advertising expenditures. Each of the competing com-
panies must make a decision as to how much money to invest in
advertising, their objective being to capture as large a portion of
the sales market as possible. The conditions necessary to examine
the problem of optimal allocation of advertising expenditures along
the lines of Game Theory have now been established.

A noteworthy attempt at modeling the problem of optimal al-
location of advertising expenditures as a problem in Game Theory
was made by Lawrence Friedman in an article published in the
Journal of the Operations Research Society of America in 1958.
In this article Friedman presented five game theory models, two

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of which were of special interest. These two models have been combined, with a few slight alterations, and are presented below as one model, which I shall call Model I.

A. MODEL I

1. Problem

Assume that Companies A and B have control of an industry. Each of the two companies is trying to obtain business from a finite number of potential customers. Both Company A and B have a fixed advertising budget, X and Y respectively, that must be allocated among the various potential customers. It is assumed that each customer's business, s_i, will go completely to the company directing the most advertising and promotion in his direction. Find an optimal advertising expenditure strategy for Companies A and B.

Let N = Total number potential customers.

Each customer, i, represents an amount of business s_i, where

i = 1, 2, 3,....N

$$\sum_{i=1}^{N} s_i = S, \text{ where S represents the total sales potential.}$$

x_i = Amount of advertising expenditures allocated by Company A to customer i.

$$\sum_{i=1}^{N} x_i = X, \quad x_i \geq 0$$

y_i = Amount of advertising expenditures allocated by Company B to customer i.

$$\sum_{i=1}^{N} y_i = Y, \quad y_i \geq 0$$

For convenience, from this point on $\sum \Rightarrow \sum_{i=1}^{N}$

The structure of the stated problem allows for the construction of the following pay off table:

Result of Advertising Expenditure Allocation	A's Payoff	B's Payoff
$x_i > y_i$	α_i	0
$x_i < y_i$	0	α_i
$x_i = y_i$	$\tfrac{1}{2}\alpha_i$	$\tfrac{1}{2}\alpha_i$

TABLE I

Since both Company A and Company B are trying to secure as large a portion of the potential sales market as possible, the difference between the two companies' sales has been chosen as the objective function. Further, Company A is assumed to be the maximizing player and Company B is assumed to be the minimizing player. Therefore if we define D, the difference between A's and B's sales, to be

$$D = \sum \alpha_i \, \text{sign}(x_i - y_i), \quad \text{WHERE} \quad D \leq S$$

$$\sum x_i = X, \quad x_i \geq 0, \quad i=1,...,N$$
$$\sum y_i = Y, \quad y_i \geq 0, \quad i=1,...,N$$

A will select a strategy to maximize D and B will select a strategy to minimize D.

The problem has now been formulated as a zero-sum, two-person game. The formulation of this game is identical to that of the well known Colonel Blotto Game.

In order to thoroughly examine the game we have developed, it is necessary to inspect two cases. In the first case both Company A and Company B have identical advertising budgets. And in the second case one of the two companies has a larger advertising

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budget than his competitor. Here we will assume that Company A's advertising budget is larger than Company B's. In summary:

Case I: Both companies have the same advertising budget: X=Y

Case II: Company A has a larger advertising budget than Company B: X>Y

For ease of presentation we will first assume a solution for both cases and then show that the assumed solution in each case is optimal.

2. Case I: X=Y

This case, where X=Y, has the property of symmetry. It corresponds very closely to the symmetric case of the Colonel Blotto Game.

The solution to this case lies in a mixed strategy for each of the two players, and the mixed strategies used by each player are identical. Note that a new term, mixed strategy, has been introduced. It has already been explained that in a game situation the players have a number of alternative actions available to them. In some cases a player would find that a pure strategy was his optimal strategy, which means he would follow the same course of action at each play of the game. On the other hand if a player's optimal strategy is a mixed strategy, then he chooses different courses of action at each play of the game. The player's choice of action would be determined by a probability distribution over his possible courses of action.

We now continue the description of Lawrence Friedman's model.

The solution for A, which is identical to the solution for B, is as follows:

Company A selects an allocation for the jth customer with equal probability from the rectangular distributions shown below.
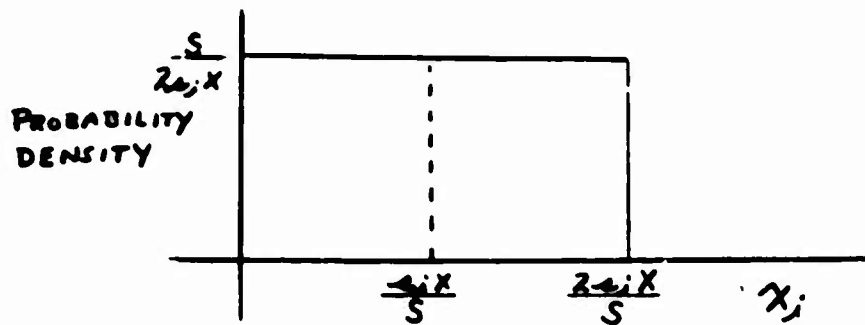
FIGURE I

This solution may be described easily in geometric terms. When described in this manner, a simple algorithm which may be used to determine both company's optimal strategy, results.

The geometric description and algorithm are now presented.

(1) Form a non-degenerate polygon the sides of which are proportioned in length to $a_1, a_2, \cdots, a_N$

THAT IS $a_1 : a_2 : \cdots : a_N = l_1 : l_2 : \cdots : l_N$

If a non-degenerate polygon cannot be formed, that is if

$$\sum_{j=1}^{\mathfrak{I}} a_j + \sum_{j=\mathfrak{I}+2}^{N} a_j < a_{\mathfrak{I}+1}$$, then all funds are allocated to $a_{\mathfrak{I}+1}$.

(2) Inscribe a circle in the polygon and erect a hemisphere upon the circle.

(3) A point is selected at random from a density uniformly distributed over the surface of the Hemisphere. This point, P, is then projected straight down on to the plane of the polygon.

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(4) The available funds are then divided in proportion to the triangular areas, a_j, subtended by P and the sides of the polygon:

i.e. $\quad x_1 : x_2 : \cdots : x_N = a_1 : a_2 : \cdots : a_N$

(5) This is a mixed strategy since P, which determines the allocations, is selected at random from a probability density, in this case the uniform density distribution.

Example

$N = 4$

$x_1 : x_2 : x_3 : x_4 = a_1 : a_2 : a_3 : a_4$
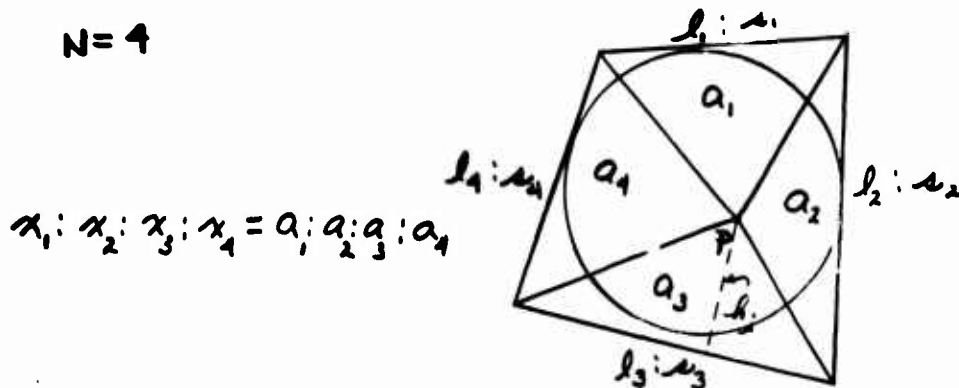
FIGURE II

3. Case II: X > Y

This case, where $X > Y$, unlike Case I, is non-symmetric. It also corresponds to a case of the Colonel Blotto Game, the non-symmetric case.

a. Solution for Company A: Allocate an amount x_j to customer j chosen at random from a rectangular distribution on the interval $(0, \frac{2a_j X}{S})$. This is identical to the strategy employed by both A and B in Case I.

b. Solution for Company B: Since B has less money to spend on advertising than A has, B cannot allocate advertising funds to each potential customer, because if he did, Company A, with the greater amount of available advertising funds, would be able to match B's efforts and if A, in addition to matching B's

efforts, allocates an additional amount $\in$, $\epsilon > 0$, to each potential customer, then A will obtain all of the potential sales. Obviously this would not be in B's best interest. In this situation B should use a mixed strategy, which would assign a probability of $\frac{Y}{X}$ of advertising to any given customer. The probability that B does not advertise to a given customer is then $(1 - \frac{Y}{X})$. To those customers to whom B does decide to allocate advertising funds, B allocates an amount y_i at random from the interval $(0, \frac{2y_i X}{S})$. (Note: B varies his strategy as to which customers he will allocate advertising funds, but to those customers to whom he does advertise, he uses the same allocating strategy as A.) The value of the game in both Case I and Case II is $D = S (1 - \frac{Y}{X})$, where in the symmetric case, Case I, D=0.

4. Proof.

Now that solutions have been assumed for each case it is necessary to show that these solutions are optimal.

a. Case I

$f_i(x_i)dx_i$ = prob (of an allocation to a given customer is between x_i, and $x_i + dx_i$ optimal strategy used.)

$F_i(x_i)$ = prob (the allocation to a given customer < x_i)

To show that the strategy given as a solution to Case I is optimal, it is necessary to show that the expected gain for A, G(Y), which is a function of $\underline{Y}$, is always ≥ 0 . $\underline{Y}$ is the vector of all possible strategies, which the opponent is allowed to employ. Now: The expected contribution to the difference in sales from customer 1

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using the strategy having cumulative density function $F_1(x_1)$ is

$$D_1 = \Delta_1[1 - F_1(y_1)] - \Delta_1 F(y_1)$$
$$= \Delta_1[1 - 2F_1(y_1)]$$

The customer will be "won" if $x_1 > y_1$. The customer will be "loss" if $x_1 < y_1$. Therefore the total expected gain as a function of the opponent's allocations $\underline{Y}$ is:

$$G(\underline{Y}) = \sum \Delta_j [1 - 2F_j(y_j)]$$

Let h_j = Altitude of the triangle of area a_j, subtended by P, (See Figure II.)

Δ_j is uniformly distributed over $(0, 2R)$, where R= radius of the inscribed circle.

so $a_j = \frac{1}{2} s_j h_j \Rightarrow a_j$ uniformly distributed over $(0, \Delta_j R)$

by definition: $X = \sum x_j = x_1 + x_2 + \cdots + x_N$

and the optimal strategy $\underline{X}$ is $x_1 : x_2 : \cdots : x_N = a_1 : a_2 : \cdots : a_N$

It follows that $x_j = \frac{a_j X}{\sum a_j}$

Therefore x_j is uniformly distributed over $(0, \frac{\Delta_j R X}{\sum a_j})$

But: $\sum a_j = \frac{1}{2}[\Delta_1 R + \Delta_2 R + \cdots + \Delta_N R]$

Therefore x_j is uniformly distributed over $(0, \frac{2 \Delta_j X}{\sum \Delta_j})$

Now: $F_j(x_j) = \int_0^{x_j} f_j(\alpha_j) d x_j = MIN\left[1, \frac{\sum \Delta_j x_j}{2 \Delta_j X}\right]$

Substituting the above into the expression for $G(\underline{Y})$ we obtain:

$$G(\underline{Y}) = \sum \Delta_j \left(1 - 2 MIN\left[1, \frac{\sum \Delta_j}{2 \Delta_j X} y_j\right]\right)$$

LET $K = \frac{\sum \Delta_j}{2 \Delta_j X}$, $K > 0$
$$y_j > 0$$

THEREFORE $MIN[1, K y_j] \leq K y_j$

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Therefore $G(\underline{Y}) \geq \sum a_i \left[1 - \frac{\sum a_i}{a_i X} y_i \right]$

$\geq \sum a_i - \sum a_i \frac{\sum y_i}{X} = S - S(\frac{Y}{X})$

But for Case I, X = Y.

SO THEREFORE $G(\underline{Y}) \geq 0$

Therefore the mixed strategy having individual distribution function $F_i(x_i)$ is an optimal minimax strategy. The geometric approach is a simple, and somewhat intutative, method of selecting N bids from a rectangular distribution such that $\sum x_i = X$

$(\text{or } \sum y_i = Y)$

b. Case II: As shown above: $F_i(x_i) = \text{MIN} \left[1, \frac{\sum a_i}{2 a_i X} x_i \right]$

and $G(\underline{Y}) \geq \sum a_i \left[1 - \frac{\sum a_i}{2 a_i X} y_i \right]$

But in this case X > Y.

Therefore $G(Y) \geq \sum a_i \left[1 - \frac{Y}{X} \right]$

$\geq S \left[1 - \frac{Y}{X} \right]$

The optimal strategy for Company B, the company with the smaller advertising budget, is to allocate funds to the the customer with probability $\frac{Y}{X}$.

Therefore, $G(\underline{X})$, the expected difference in sales if Company A uses the strategy vector $\underline{X}$ and Company B uses the strategy given above, is: $G'(\underline{X}) = \sum \left[a_i F_i(x_i) (\frac{Y}{X}) - a_i (1 - \frac{Y}{X}) F_i(x_i) \right]$

$= \sum a_i \left[2 F_i(x_i) (\frac{Y}{X}) - 1 \right]$

Where $F_j^1(y_j)$, the cumulative distribution function for the jth allocation for Company B is

$$F_j^1(y_j) = \text{Min}\left[1, \frac{\sum a_j}{2a_j \cdot X} y_j\right]$$

and so $G(\underline{X}) = \sum a_j\left[2\left(\frac{Y}{X}\right) \quad \text{Min}\left(1, \frac{\sum a_j}{2a_j \cdot X} y_j\right) - 1\right]$

$$\geq \sum a_j\left[2\left(\frac{Y}{X}\right) \frac{\sum a_j}{2a_j \cdot X} x_j - 1\right]$$

$$\geq \sum a_j\left[\frac{Y}{X} - 1\right]$$

and finally $G(\underline{X}) \leq S\left[1 - \frac{Y}{X}\right]$

This implies that the maximim solution for competitor A equals the minimax solution for competitor B, and the value of the game, which here is the difference in sales, is:

$$D = S\left[1 - \frac{Y}{X}\right]$$

This completes the description of Mr. Friedman's game theory model for the determination of optimal advertising expenditure allocations.

Both Friedman's model and the models which will be discussed below are extensions of the classic Colonel Blotto Game which was mentioned at various times in the preceeding discussion. Since the Colonel Blotto Game plays such an important role in the models examined in this paper it would be well to present a brief description of that game at this point.

The Colonel Blotto Game is seen in many forms, but most of these forms are basically the same. Melvin Dresher in his book Games of Strategy, Theory and Applications , gives a simple, but

very complete, description of the basic Colonel Blotto Game and its structure. Here is Dresher's description, modified slightly in an effort to generalize the description.

> "Colonel Blotto and his enemy each try to occupy two posts by properly distributing their forces. Let us assume that Colonel Blotto has X regiments and his enemy has Y regiments which are to be divided between the two posts. Define the payoff to Colonel Blotto at each post as follows: If Colonel Blotto has more regiments than the enemy at the post, Colonel Blotto receives the enemy's regiments plus one (the occupation of the post is equivalent to capturing one regiment); if the enemy has more regiments than Colonel Blotto at the post, than Colonel Blotto at the post, then Colonel Blotto loses one plus his regiments at the post; if each side places the same number of regiments it is a draw and each side gets zero. The total payoff is the sum of the payoffs at the two posts." [1]

In this description Dresher mentions only two objectives, obviously there could be any number of objectives which the opposing sides are trying to capture. In our advertising expenditure models the objectives are the units of business potential customers provide to competing companies. If Colonel Blotto's forces, and his enemy's forces, are infinitely divisible, then the game is said to be continuous. On the other hand if the opposing forces are not infinitely divisible and must be considered in discrete units, then

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[1] Dresher, M., Games of Strategy, Theory and Applications, Prentice-Hall Inc., Englewood Cliffs, N.J., 1961, Pg. 7

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it is said that the game is discrete. Friedman's model is an extension of the discrete case of the general Colonel Blotto Game.

The following models initially consider extensions of the continuous case of the general Colonel Blotto Game. It will be seen that the results of this analysis are generally the same as those arrived at by Friedman.

5. <u>Criticism of Model I and Background for Model II</u>.

Considering Friedman's stated and implied assumptions, the preceeding model is sound and provides a basis for further investigation into the application of game theory to the problem of optimally allocating advertising expenditures. However, in its present form Friedman's model is little more than an academic exercise since its application to the real world is limited to a few special cases. It is felt that Friedman's model could be improved in three areas. First, Friedman's model is not as streamlined mathematically as it could be. Secondly, his assumptions concerning the effectiveness of each company's advertising dollar is generally not valid. And finally, the use of the difference in sales as an objective function, or measure of effectiveness, does not reflect the thinking of many managers in business today. The remaining portions of this paper will be devoted to an attempt at rectifying the three problem areas just mentioned.

Although Friedman's development provides a rather simple algorithm for use in determining the actual amount of money to be spent on each potential customer, or group of customers, it seems that it is not necessary to appeal directly to analytic geometry to show that under his assumptions his solutions are

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optimal. M. P. Peisakoff, in a paper written in 1951 for the Rand Cor-
poration, examined the general case of the Colonel Blotto Game. By
looking at the two-person Colonel Blotto Game in its most general form
Peisakoff was able to cover both the symmetric and non-symmetric cases
without separating them as Friedman did. In his paper, Peisakoff proves
very concisely that optimal strategies do indeed exist for both players
in the general case of the Colonel Blotto Game. Further the solutions
which Peisakoff suggests are identical to the ones which Friedman used
as a basis for the development of his model. This suggests that it
is possible to modify Peisakoff's work and apply it to the problem
under study in this paper. The hope here is to present a model which
is less cumbersome than Friedman's, and which does not appeal directly
to analytic geometry. Further, where as Friedman developed his model
based on the discrete Colonel Blotto Game, the model presented below
is based on the continuous, and more general, case of the Colonel
Blotto Game.

The following is an advertising expenditure allocation model
based on a modification of Peisakoff's theorem and subsequent proof.

B. MODEL II

As in Model I we have to companies, A and B, both of whom are
trying to obtain business from a group of potential customers.

Let X = Company A's total available advertising budget.

Y = Company B's total available advertising budget.

S = Total potential amount of sales.

Company A selects a function x from the non-negative measurable
functions on the interval $[0,S]$ such that $\int_0^S x(t)\, dt = X$, $X > 0$

$x(t)$ = portion of available advertising funds allocated by
Company A to some given potential customers or group
of customers.

Company B selects a function y from the non-negative measurable function
on the interval $[0, 1]$ such that $\int_0^1 y(t)\,dt = Y$, $Y > 0$

$y(t)$ = portion of available advertising funds allocated
by Company B to some given potential customer or
group of customers.

It is assumed that Company A is the maximizing player and Company B
is the minimizing player. Therefore by convention, Company A re-
ceives the "winnings." In this model Company A receives a net gain
over B of

$$\int_0^1 \frac{s(t)}{S} \operatorname{sign}\left[x(t) - y(t)\right] dt \quad , \text{ where } \frac{s(t)}{S} \text{ is}$$

the fraction of the total sales represented by a given customer or
group of customers.

$s(t) \geq 0$

$$\int_0^1 \frac{s(t)}{S}\,dt = \frac{1}{S}\int_0^1 s(t)\,dt = 1 \quad , \text{ therefore } \int_0^1 s(t)\,dt = S$$

Now assume that $X \geq Y$ since the case of $Y > X$ can be gotten by sim-
ply interchanging players (i.e. companies).

1. **Definition of Strategies**

A strategy for Company A is a random function $x_\alpha(t)$ which is
measurable on the Product $A \times [0, 1]$.

α is a random variable on the arbitrary space A.

and $\int_0^1 x_\alpha(t)\,dt = X$ for all α

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Likewise a strategy for Company B is a random function $y_\beta(t)$ which is measurable on the Product Bx $[0, 1]$.

β is a random variable on the arbitrary space B.

and $\int_0^1 y_\beta(x)\,dt = Y$ fall all β.

2. Theorem

Suppose that for a strategy $x_\alpha(t)$, for each fixed t, the random variable $x_\alpha(t)$, is uniformly distributed on the interval $\left[0, \frac{2a(t)X}{S}\right]$. Suppose further that for a strategy $y_\beta(t)$, for each fixed t, the random variable $y_\beta(t)$ is uniformly distributed $\left[0, \frac{2a(t)X}{S}\right]$ with probability $\frac{Y}{X}$, and is zero with probability $\left[1 - \frac{Y}{X}\right]$.

Then x_α is an optimal strategy for Company A and y_β is an optimal strategy for Company B. Further the value of the game is $S\left[1 - \frac{Y}{X}\right]$.

3. Proof

Let μ be the measure associated with α

σ_t be the uniform distribution on $\left[0, \frac{2a(t)X}{S}\right]$.

$y(t)$ be fixed

Applying Fubini's theorem and the assumption of this model we obtain:

$$\int_A \mu\,d(\alpha) \int_0^1 \text{SIGN}\left[x_\alpha(t) - y(t)\right]\left(\frac{a(t)}{S}\right)\,dt =$$

$$= \int_0^1 dt\left(\frac{a(t)}{S}\right)\left[\int_{y(t)}^\infty \sigma_t\,d(\phi) - \int_{-\infty}^{y(t)}\sigma_t\,d(\phi)\right]$$

Let $a'(t) = \frac{a(t)}{S}$, $\int_0^1 a'(t)\,dt = 1$

$$= \int_0^1 dt\,a'(t)\left[2\int_{y(t)}^\infty \sigma_t\,d(\phi) - 1\right]$$

$$= \int_0^1 dt\,(x)^{-1}\,\text{MAX}\left[0, 2a'(t)X - y(t)\right] - 1$$

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Now let $T = \{t \mid 2\alpha'(t)X < y(t)\}$ then

$$= (X)^{-1} \int_{[0,1]-T} dt \, [2\alpha'(t)X - y(t)] - 1$$

$$= (X)^{-1} \left\{ 2X - Y - \int_T [2\alpha'(t)X - y(t)] dt \right\} - 1$$

$$= \left\{ 2 - \frac{Y}{X} - \frac{1}{X} \int_T [2\alpha'(t)X - y(t)] dt \right\} - 1$$

$$= \left\{ 1 - \frac{Y}{X} \right\} + \frac{1}{X} \int_T [-2\alpha'(t)X + y(t)] dt$$

$$\geq \left\{ 1 - \frac{Y}{X} \right\}$$

Since $Y \leq X$ and $X, Y \geq 0$ then

$$0 \leq \left\{ 1 - \frac{Y}{X} \right\} \leq 1$$

It has now been shown that if Company A employs strategy $\alpha(t)$ against any fixed strategy $y(t)$ of Company B, he will be assured of receiving $\left\{ 1 - \frac{Y}{X} \right\}$. But the objective function has been defined as the difference between A's and B's sales. Therefore the quantity $\left\{ 1 - \frac{Y}{X} \right\}$ is that fraction of the total potential sales which equals A's net gain over B. So if A employs strategy $\alpha(t)$ he will expect to receive at least $S\left\{ 1 - \frac{Y}{X} \right\}$ more sales than B.

Therefore if strategy $\alpha(t)$ is used

$$D \geq S\left\{ 1 - \frac{Y}{X} \right\}$$
, where D = the difference between A's and B's sales

and D goes to A.

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Now: Let the measure associated with A be ν and let $p_* = (\frac{y}{x})\varsigma + (1-\frac{y}{x})$ U, where U assigns probability 0 to 1. Then given a fixed $\alpha(t)$

$$\int_B \nu \, d\beta \int_0^1 \text{SIGN}\{\alpha(t) - \eta_\beta(t)\} \alpha'(t) \, dt =$$

$$= \int_0^1 dt(\alpha'(t)) \left[\int_{-\infty}^{\alpha(t)^-} p_t \, d\phi - \int_{\alpha(t)^+}^{+\infty} p_t \, d\phi \right]$$

$$= \int_0^1 dt \, \alpha'(t) \left[1 - 2(\frac{y}{x}) \int_{\alpha^+(t)}^{\infty} p_t \, d\phi \right]$$

$$= 1 - \int_0^1 dt \, (\frac{y}{x^2}) \, \text{MAX}\, [0, 2\alpha'(t)x - \alpha(t)]$$

Now let $T' = \{t \mid 2\alpha'(t)x < \alpha(t)\}$ then

$$= 1 - (\frac{y}{x^2}) \int_{[0,1]-T'} dt [2\alpha'(t)x - \alpha(t)]$$

$$= 1 - (\frac{y}{x^2}) \left\{ 2x - x \int_{T'} dt [2\alpha'(t)x - \alpha(t)] \right.$$

$$= \left\{ 1 - \frac{y}{x} \right\} - \left\{ \frac{y}{x^2} \int_{T'} dt [\alpha(t) - 2\alpha'(t)x] \right\}$$

$$\leq \left\{ 1 - \frac{y}{x} \right\}$$

Therefore if Company B uses the strategy $\eta_\beta(t)$ against any fixed strategy of A, then Company B can expect that A will receive no more than $\left\{ 1 - \frac{y}{x} \right\}$. Again $\left\{ 1 - \frac{y}{x} \right\}$ is that fraction of the total possible sales equivalent to the difference vetween A's and B's sales. Therefore B can insure that A will not be able to obtain greater than $S\left\{ 1 - \frac{y}{x} \right\}$ sales more than B. If B employs strategy $\eta_\beta(t)$

$$D \leq S\left\{ 1 - \frac{y}{x} \right\} \text{ , where D = difference between A's and B's sales}$$

and D goes to A.

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In summary:

$G(x(t), y(t))$ = Payoff function, objective function

$X(t)$ = Any possible strategies employed by Company A

$x_\alpha(t)$ = Company A's optimal strategy

$y(t)$ = Any possible strategy employed by Company B.

$y_\beta(t)$ = Company B's optimal strategy.

$$G(x(t), y_\beta(t)) \leq G(x_\alpha(t), y_\beta(t)) \leq G(x_\alpha(t), y(t))$$

$$G(x_\alpha(t), y_\beta(t)) \quad = \text{value of the game}$$

$$= S\left\{1 - \frac{Y}{X}\right\}$$

The advertising allocation model just described is obviously more streamlined and general than Friedman's, and it arrives at the same results as did Friedman's model. The most important of these results perhaps being that given two Companies A and B, with total advertising budget X and Y respectively, and $X > Y$, the difference, D, between A's and B's sales is:

$$D = S\left\{1 - \frac{Y}{X}\right\} \qquad \text{, where S is the total potential sales.}$$

However this second model lacks a simple means of determing exactly how to distribute Company A's and Company B's advertising budgets. Friedman's model on the other hand provides an easily followed algorithm which can be used to decide how much of the advertising budget to allocate to a customer, or group of customers,   .
But this possibly is not too important, since management is more often interested in the total expected results of an advertising campaign costing Z number of dollars. That is, the management of

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large companies, for example Sears Roebuck, General Motors, Goodyear and Dupont, feel that through market research they can determine how to spend their advertising money. What they want to know is how much money to spend and given a specific amount of money to spend, what can they expect to receive back in terms of sales. Both Friedman's model and the model just presented based on Peisakoff's work can help to answer these questions. Perhaps from a strictly mathematical point of view the second model is more appealing, since it is first very general, and second, less cluttered than the first.

IV. <u>RELATIVE EFFECTIVENESS OF ADVERTISING DOLLARS</u>

A. DISCUSSION

Models can be useful for two purposes, first to predict the outcome of the real world events they represent and second to establish a firm structure for the operation under consideration. A model may at the same time be able to do one of these and not the other. The two models presented so for are an example of this, as seen in the following discussion.

The models do provide some interesting points concerning the structure of the problem. They tell us that in general optimal pure strategies do not exist. For the case of a small company in competition with a larger one, the models say that the smaller company should disregard at random certain portions of the potential market and meet the competition on equal terms in the remaining portions of the market. Of course if the optimal mixed strategies are to be indeed optimal in the real world then it is necessary that management keep their allocation decisions secret until they actually activate their advertising programs.

It should be obvious from the above discussion that by just merely setting up our models we can gain insight into the operation under study. Now for the question of whether or not Models I and II are of any use in predicting the outcome of real world events. First of all it should be noted that it is not necessary that models be absolutely accurate in their predictions, but it is necessary that their predictions agree closely enough with reality so as not to cause false decisions to be made by those persons employing the models.

From this point of view both of the preceeding models, in their present forms, are unacceptable. The validity of the predictions made by models is highly dependent on the validity of the various assumptions upon which the models are built. If these assumptions are not valid, then it is highly unlikely that the models under study will be able to give meaningful answers to real world problems.

Throughout the development of both Friedman's model and the second model based on Peisokoff's work, it was assumed that each dollar spent on advertising by one company was as effective as each dollar spent on advertising by the second company. Another way of stating this is that it was assumed that the amount of business which one company's advertising dollar could buy was equal to the amount of business that the other company's advertising dollar could buy. However, it is quite easy to find real world examples which show that this assumption is completely invalid. Because this assumption is not valid the use of the equations given in the models to determine the difference between the companies' sales may result in very erroneous calculations, as shown in the examples below.

All of the data in the following examples has been taken from the 26 August 1968 issue of Advertising Age and this data is for the year 1967.

1. Example 1

In this example we shall look at General Electric and Radio Corporation of America. These two companies account for greater than seventy-five percent of the total sales of domestically produced appliances, televisions, and radios.

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	Total Advertising Expenditures (in #)	Total Sales (in $)
GE	7.8×10^7	7.7×10^9
RCA	7.5×10^7	3.0×10^9
Sum of G.E.'s and RCA's sales		10.7×10^9

TABLE II

Let D' = difference in the sales based on actual figures

= (G.E.'s sales) - (RCA's sales)

$D' = 4.7 \times 10^9$ dollars

But according to the equation developed in the two models presented in this paper:

D = Calculated difference in sales between two companies

$= S \left\{ 1 - \frac{y}{x} \right\}$ where   X = G.E. advertising expenditures

$= 10.7 \times 10^9 \left\{ 1 - \frac{7.5 \times 10^9}{7.8 \times 10^9} \right\}$ Y = RCA advertising expenditures

$= 10.7 \times 10^9 \left\{ 1 - 0.962 \right\}$ S = Total sales in dollars

$= 10.7 \times 10^9 \left\{ .038 \right\} = \left\{ .0380 \times 10^9 \right\}$

$D = 3.80 \times 10^7$

Therefore $D \neq D'$

$D' - D > 1 \times 10^2$

The difference between the actual and calculated results, i.e. $D' - D$ is obviously too great to be ignored. It is obvious in this example that General Electric's advertising dollar was much more effective, in terms of dollars of sales per dollar of advertising, than Radio Corporation of America's advertising dollar.

2. Example 2

In order to show that the preceeding example is not a unique situation, the case of Eastman Kodak vs. Polaroid Corporation will be examined. These two companies control greater than ninenty percent of the sales of domestically produced photographic equipment.

	Total Advertising Expenditures (in $)	Total Sales of Each Company (in $)
Kodak	4.9×10^7	1.7×10^9
Polaroid	2.1×10^7	0.37×10^9
Total Sales,		2.07×10^9

TABLE III

Let $D' =$ difference in sales based on actual figures

$\quad\quad =$ (Kodak's sales) - (Polaroid's sales)

$D' = 1.33 \times 10^9$ dollars

But according to the equation developed in the two models:

$D =$ calculated difference in sales

$\quad = S\left\{1 - \dfrac{Y}{X}\right\}$, where $\quad X =$ Kodak's advertising funds

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad Y =$ Polaroid's advertising funds

$\quad = 2.07 \times 10^9 \left\{1 - \dfrac{2.1 \times 10^9}{4.9 \times 10^9}\right\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad S =$ Total sales

$\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (all the above in dollars)

$\quad = 2.07 \times 10^9 \left\{1 - 0.428\right\}$

$\quad = 2.07 \times 10^9 \left\{0.572\right\} \quad = \quad 1.18 \times 10^9$

$D \doteq 1.18 \times 10^9$

$D' - D = 1.5 \times 10^8$ dollars

Again the estimated and actual value of D differ by a noticeable
amount. It appears, in this example that Kodak's advertising dollar is
more effective than Polaroid's.

B. RELATIVE ADVERTISING EFFECTIVENESS OF DOLLARS

If the models discussed in this paper are to be of any use it is
necessary to take into account the relative effectiveness of the ad-
vertising dollars of the companies involved. Again take the case of
General Electric and Radio Corporation of America. For every dollar
spent by G. E. on advertising approximately 100 dollars in sales
were realized. But for every dollar spent by RCA on advertising
only 40 dollars in sales were returned. In other words each of G.E.'s
advertising dollars returned 2.5 times the quantity of sales that each
advertising dollar of RCA's did.

When it is said that Company A's advertising dollar is more ef-
fective than Company B's, it is necessary to understand exactly what
is meant by advertising effectiveness, and why one company's adver-
tising dollar is more effective than another company's.

In answer to the first question it might be more proper to talk
of relative advertising effectiveness. If we let the number of dol-
lars of sales per dollar of advertising be a measure of the advertising
effectiveness of each company's advertising dollar, then the ratio of
two companies' dollar advertising effectiveness will result in the
relative effectiveness of the advertising dollars of the two com-
panies involved.

That is:

> Let A = Company A's total sales in dollars
>
> B = Company B's total sales in dollars
>
> X = Company A's advertising budget
>
> Y = Company B's advertising budget

then:

$$\frac{A}{X} = \text{Advertising effectiveness of Company A's advertising dollar}$$

$$= \mathcal{E}_A$$

$$\frac{B}{Y} = \text{Advertising effectiveness of Company B's advertising dollar}$$

$$= \mathcal{E}_B$$

If $\mathcal{E}_A \leq \mathcal{E}_B$

then: $\dfrac{\mathcal{E}_A}{\mathcal{E}_B} = \text{Relative advertising effectiveness}$

$$= \mathcal{E}_R$$

If $\mathcal{E}_B < \mathcal{E}_A$

then $\mathcal{E}_R = \dfrac{\mathcal{E}_B}{\mathcal{E}_A}$

1. Acceptance

It is fairly easy to answer the question of why one company's advertising dollar is more effective than another company's advertising dollar. This answer is in two parts. First of all the concept of acceptance is extremely important. This concept seems not to have a specific definition, but a good synonym would be reputation. A company which has a good reputation does not need to spend as much money on advertising to attain a given quantity of sales as it would if its reputation was not as good. Reputation is built on the quality of a company's products and services. If a company produces

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high-quality goods and/or supplies effective and efficient services
then it can be assured of a good reputation. Reputation, or acceptance
as certain executives prefer, is a source of "free" advertising, which
may or may not be good. "Free" advertising consists mainly of reviews
of a company's products and services found in various publications,
and word of mouth, that is a satisfied customer, or a dissatisfied
customer, expressing his opinion of a company to another potential
customer. Obviously if the reviews are good and the customers are
satisfied, then a company needs to spend less money to convince po-
tential customers to purchase that company's products and/or services.
On the other hand if the reviews are not good and customers are un-
happy with the company's products and/or services, then the company
is forced to spend additional money in an attempt to overcome the
ill effects of this unfavorable "free advertising." The effects
of reputation or acceptance are so strong that many companies find
it to their financial advantage to spend extra money on quality con-
trol in order to build a good reputation

The concept of acceptance also takes into account the length
of time a company has been in existence The longer a company has
been established, the less effort it takes to make the public aware
of its presence. This effort is reflected in the amount of adver-
tising done by the company A new company in an industry, unlike
one of the older, well-established ones, must spend a good deal of
money on advertising just to get their name in the minds of potential
customers. It is possible that a large percentage of this initial
advertising effort by young companies is unproductive. Unproductive
in the sense that M dollars spent by young companies on advertising
results in a lesser amount of sales than an older company would

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realize with an equal expenditure of funds for advertising. Customers often tend to be conservative in their buying habits and feel more comfortable dealing with an older, better known company. Once a company has been accepted as an established element of an industry, then it does not need to spend a great deal of money on advertising just to let customers know that it is around. At this point the company can now divert its advertising effort to specific areas of merchandise.

2. Non-optimal Strategies

The second part of the answer as to why one company's advertising dollar is apparently more effective than another company's may lie in the selection of advertising strategies. If the companies involved in the conflict situation as outlined in this paper do not pick optimal strategies, then the value of the game, $D = S\left(1 - \frac{Y}{X}\right)$ will not be realized. If one company selects an optimal, or near optimal strategy and the other company selects a non-optimal strategy, then it will appear that the company employing the optimal strategy has the more effective advertising dollar. The company using an optimal advertising strategy will in fact possess advertising dollars which are relatively more effective than the company not employing an optimal strategy, since according to the Theory of Games, players in a game situation cannot maximize their utility by using non-optimal strategies.

In mathematical symbols the above can be stated more clearly as follows:

Recall from the development of the advertising model based on Piesakoff's work that:

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$X(t)$ = Any possible strategy employed by Company A

$X_\alpha(t)$ = Company A's optimal strategy

$y(t)$ = Any possible strategy employed by Company B

$y_\beta(t)$ = Company B's optimal strategy

D = Value of game.

$$G[X(t), y(t)] = S\left\{1 - \frac{y(t)}{X(t)}\right\}$$

= objective functions, when optimal strategies employed equals value of game.

$$G[X(t), y_\beta(t)] \le G[X_\alpha(t), y_\beta(t)] \le G[X_\alpha(t), y(t)]$$

If A uses an optimal strategy, but B does not, the value of the objective function is greater than the value of the game. i.e. If the value of the game is $D = S\left\{1 - \frac{y}{X}\right\}$ then if D' is the value of the objective function given A uses an optimal strategy and B does not, we obtain $S\left\{1 - \frac{y}{X}\right\} = D < D'$.

On the other hand if B uses an optimal strategy and A does not, then the valu of the objective function, D'', is less than the value of the game, i.e.

$$D'' < D = S\left\{1 - \frac{y}{X}\right\}$$

In the example of General Electric versus Radio Corporation of America it was noted that $S\left\{1 - \frac{y}{X}\right\} = D < D'$ and it was said that General Electric's advertising dollars appeared relatively more effective than R.C.A.'s We can conclude that since $S\left\{1 - \frac{y}{X}\right\} = D$ AND $D < D'$, RCA, the company replacing Company B in the model, is not employing an optimal advertising strategy. Even if RCA, in this example, is distributing its advertising effort, based on the absolute value of the advertising funds available, in an optimal

manner, it is possible that the relative effectivenesss of RCA's advertising dollar has been degraded by a lack of "acceptance." Therefore, even though Y dollars are available in absolute terms, these Y dollars may be only as effective as $\mathcal{A}Y$ dollars, $0 \leq \mathcal{A} \leq 1$, as a result of the influence of "acceptance." In general $S\left\{1 - \frac{Y}{X}\right\} \neq S\left\{1 - \frac{\mathcal{A}Y}{X}\right\}$, this is another very likely source of error in the calculation of D.

We have seen that the effectiveness of a company's advertising dollar may be degraded by an improper choice of advertising strategies and/or by the ill effects of that somewhat abstract factor called acceptance. It may also be degraded by a lack of secracy. If a given company can find out in advance exactly how his competitors are going to allocate their funds, then it is possible to adjust his advertising program to take full advantage of this knowledge. It is therefore essential, especially for a company with a small advertising budget, to keep all allocation schemes secret until they have been put into effect.

Assuming secracy , it is now necessary to specify a method by which our models may be used in order to be able to predict, with some accuracy, the outcome of real world events. For now the real world events of interest will be the difference in sales between two companies given the companies' advertising budgets.

C. MODIFICATIONS OF MODELS I AND II TO ALLOW FOR THE RELATIVE EFFECTIVENESS OF ADVERTISING DOLLARS

Obviously it would be extremely difficult, if not impossible, to separate and quantitize the effect of acceptance non-optimal strategy selection which result in incorrect calculations of the value of D for a given pair of companies for a given period of time in a real world situation. If it were possible to determine

exactly which non-optimal strategy a company elected to employ, then, with a knowledge of the effects of acceptance on the relative effectiveness of the company's advertising dollar, it would be possible to predict a correct value for D. However, considering the general case, it is possible to have an infinite number of non-optimal strategies to choose from. The following is a suggested approach to circumvent the difficulties just described and to establish a method of correctly predicting the value of D.

(1) The companies in question must be examined in pairs.

(2) It is assumed that the effect of acceptance are constant in the short run, specifically over approximately a two year period.

(3) It is assumed that companies do not radically alter advertising allocation schemes from one year to the next.

(4) It is assumed that for each pair of companies of interest it is possible to obtain the following data:

    (a) Total sales.

    (b) Total advertising expenditures.

(This data is generally available.) It is further assumed that if this is year X, the required data is available at least up to year X-1 (last year).

(5) If $A' =$ Total sales in dollars for Company A

    $B' =$ Total sales in dollars for Company B

    $X' =$ Company A's total expenditures for advertising

    $Y' =$ Company B's total expenditures for advertising

A', B', X', and Y' are based on data for year X-1.

Let $D' = A' - B'$

    $S' = A' + B'$

$$D' = N\ S'\left\{1 - \frac{Y'}{X'}\right\}$$

N is a constant which takes into account the effect of non-optimal strategy selections and acceptance

$$N = \frac{D'}{S'\left\{1 - \frac{Y'}{X'}\right\}}$$

N is determined for year X-1 and is used to predict D for year X. The only available information concerning year X are estimates for X, Y, and S, the potential value of the market.

$$D = N\ S\left\{1 - \frac{Y}{X}\right\}$$

$$\text{where}\quad N = \frac{D'}{S'\left\{1 - \frac{Y'}{X'}\right\}}$$

Example

Company	Total Adv. Expenditure	Total Sales
A	50	200
B	30	90

DATA FOR YEAR X-1

TABLE IV

$$S' = 200 + 90 \qquad X' = 50$$
$$= 290$$
$$D' = 200 - 90 \qquad Y' = 30$$
$$= 110$$

$$D = NS'\left\{1 - \frac{Y'}{X'}\right\}$$

$$110 = N(290)\left\{1 - \frac{2}{3}\right\} \qquad \text{SO THEREFULL} \qquad N = \frac{550}{580}$$

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Example (continued)

Company	Total Adv. Expenditures	Total Sales
A	100	400
B	60	180

DATA FOR YEAR X

TABLE V

Let D^* = Actual difference between A and B's sales

$\quad$ = 400 - 180

$D^* = 220$

Let D = Calculated value of the difference between A and

$\quad$ B's sales.

$D = N S\left\{1 - \dfrac{Y}{X}\right\}$ $\quad$, In a real situation S, Y, and X

$\qquad N = \dfrac{550}{580}$ $\qquad$ would most likely be estimates.

$\qquad S = 400 + 180$

$\qquad = 580$

$\qquad Y = 60$

$\qquad X = 100$

So $D = \left(\dfrac{550}{580}\right)(580)\left\{1 - \dfrac{60}{100}\right\}$

$\qquad = 550\left\{\dfrac{2}{5}\right\}$

$D = 220$

Therefore $D = D^*$

In summary, the preceeding is a method which uses past information to predict the future difference in sales between two companies. The past data is used specifically to determine a constant N which

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accounts for variations from the ideal caused by non-optimal strategy selections on the part of the companies involved, and the effects of acceptance on each company's advertising dollar effectiveness. Once N is determined it is used as a correction factor in the equation originally developed to predict the difference in sales between two companies. That is

$$D = N \ S\left\{1 - \frac{Y}{X}\right\}$$

The modified equation will give accurate predictions only as long as assumptions (2) and (3) mentioned above are true. In general these assumptions are true only in the short run, therefore N should be continually updated whenever practical.

D. USE OF MODEL IN DETERMINATION OF TOTAL SALES EFFECT.

1. Profit

So far we have dealt only with differences in sales which can be expected if two companies allot X and Y dollars respectively to advertising. However business executives do not find this to be an acceptable criterion by which to judge the effectiveness of their advertising efforts. The prime concern of management is profit, pure and simple.

Most managers will sacrifice sales in favor of profit. Profit, for some reason, has a soothing effect on such easily ruffled individuals as stock holders, and members of the board of directors. In view of this extremely strong tendency of large business to judge the effectiveness of advertising by profits returned, it seems advisable to develop an expression which will be able to predict profit as a function of advertising expenditure.

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Let A = Company A's sales

        X = Company A's advertising expenditure

        B = Company B's sales

        Y = Company B's advertising expenditures

        D = Difference between A and B's sales

        S = Total sales

$$A + B = S \qquad D = NS\left\{1 - \frac{Y}{X}\right\}$$

Therefore,

$$B = S - A$$

$$D = A - B = NS\left\{1 - \frac{Y}{X}\right\}$$

But $B = S - A$

So $A - (S - A) = NS\left\{1 - \frac{Y}{X}\right\}$

$$2A - S = NS\left\{1 - \frac{Y}{X}\right\}$$

$$2A = NS\left\{1 - \frac{Y}{X}\right\} + S$$

$$= S\left[N\left\{1 - \frac{Y}{X}\right\} + 1\right]$$

$$A = \frac{S}{2}\left[N\left\{1 - \frac{Y}{X}\right\} + 1\right]$$

Therefore Company A's sales $= \frac{S}{2}\left[N\left\{1 - \frac{Y}{X}\right\} + 1\right]$

Now   Profit = Sales (in dollars) - Cost

           Let P = profit

             C = costs

$$P = \frac{S}{2}\left[N\left\{1 - \frac{Y}{X}\right\} + 1\right] - C$$

The above expression gives the predicted profit for Company A.
B's predicted profit may be found in a similar manner.

V.   MARKETS WITH N - COMPETITORS

Although the models, and model modifications, presented thus far
in this paper appear to be potentially very useful, they have one major
flaw which up to this point has not been explicitly mentioned. The
general formulazation of the models has been along the lines of a zero-
sum, two-person game. That is, it has been tacitly assumed that the
companies involved in the conflict situation previously described at-
tribute the same value of utility to every dollar. Specifically it has
been assumed that each company's utility function is linear in dollars
and identical. It has further been assumed that the companies of
interest have been members of a duopoly; perhaps this last assumption
is the most difficult to justify. Obviously very few actual industries
are controlled entirely by two companies. In general an industry is
composed of many individual companies. Considering this situation, it
would probably be advisable to restructure the problem of optimal ex-
penditure allocations along the lines of an N-person game, $N > 2$,
but this presents some new problems.

The theory of games has not been advanced to the point of being
able to solve the general N-person game, except for a very few special
cases. It has proven exceedingly difficult to establish methods for
dealing with certain situations which can only arise when more than
two parties are involved in a conflict. These situations involve
collusion on the part of two or more players. For example, it is
possible for certain players, if allowed by the rules of the game,
to form alliances and then establish jointly optimal strategies.
These jointly optimal strategies could be such that they would be

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non-optimal if employed by any one of the colluding players. Further the rules of the game could allow players to switch from one coalition to another between plays of the game. The complexity of handling N-person games should be apparent from this brief discussion of the subject.

Although the problem of optimally allocating advertising expenditures has not been analyzed as an N-person game in this paper, the models which have been discussed may still be useful in industries other than duopolies. First of all it has been stated that one of the biggest difficulties encountered in N-person games, $N > 2$, is the collusion which is possible on the part of the players. However if collusion is disallowed by the rules of the game, then analysis of N-person games becomes considerably less difficult. In the United States it is unlawful for corporations to engage in collusion with a view towards reducing competitors. Therefore it can be assumed that businesses can not, in general, form coalitions and formulate joint advertising strategies. This implies that each company must act independently in the area of advertising. In view of these considerations the following methods are suggested for the implementation of the models discussed in this paper by companies competing in non-duopoly, or monopoly industries.

(1) A company called A, with a given advertising budget may wish to estimate its sales, or profit, relative to each of its competitors individually. This may be accomplished simply by equating the value of S, sale potential, to the estimated sales for which Company A and any one of its competitors together could account for. The value of X and Y are as described previously. Company A may use this technique for each competitor.

(2) A company called A, with a given advertising budget may wish to estimate its sales, or profit, relative to all of its competitors taken in aggregate. Here, although there may be many individual players, i.e. competitors, the game is reduced to a two-person game where one player is an individual company, in this case A, and the other player is a composite player formed by the sum of the remaining companies. More simply, the two players in this game are Company A and everyone else. The quantity S is the estimated total potential sales of the industry. Company A has an advertising budget of X dollars. The opposing player has an advertising budget of Y" dollars. Where $Y" = \sum_{j=2}^{n} Y_j'$, $Y_j \geq 0$, $Y_j' =$ advertising budget of Company i , (index starts at 2 since Company A is not included) Company i is one of A's competitors forming the composite player.

Then:

$$D = NS\left\{1 - \frac{Y"}{X}\right\} \quad , \text{ if } Y" \leq X$$

$$= NS\left\{1 - \frac{X}{Y"}\right\} \quad , \text{ if } Y" > X$$

$$= \frac{S}{2}\left[N\left\{1 - \frac{Y"}{X}\right\} + 1\right] - C , \text{ if } Y" \leq X$$

$$= \frac{S}{2}\left[N\left\{1 - \frac{X}{Y"}\right\} + 1\right] - C , \text{ if } Y" > X$$

The above two methods for dealing with industries which are not duopolies may or may not give accurate predictions, and are only suggested methods, with any further development left to future research,

VI. CONCLUSION

An attempt has been made in this paper to present useful models, based on Game Theory, deal'ng with the problem of optimally allocating advertising expenditures. If the models and relations discussed in this paper are applied to real world situations it is felt that the management of a given company will be able to accurately predict, over the short run, the company's sales behavior relative to any given competitor. However, since many of the equations developed in this paper are used with estimated quantities and data from previous time periods, it is necessary to be aware of the possibility of inaccurate predictions caused by the use of poor input data. No model is able to correct for poor data, unless of course the error in the data is fairly constant.

It must be remembered that the problems of developing effective advertising policies are very complicated and one must not think that these problems can be solved by merely substituting appropriate terms into a set of handy equations.

It was not the purpose of this paper to present an absolute method which would solve any advertising allocation problem. There are many parts of the models presented in this paper which could be improved through future research. The application of game theory to the problem of advertising funds allocation seems to have considerable merit, and even the crude models presented and discussed throughout this paper appear to have at least limited use.

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13. ABSTRACT

This paper looks at the problem of optimally allocating advertising funds from a game theory point of view. Two basic models are presented and then expanded upon. These models are simply structured having originated from work done on the Colonel Blotto Game in the early 1950's. A prime objective of this paper is to briefly review representative examples of work previously done in this area and indicate the possible direction of future research. One of the more insteresting extensions of the basic model is the development of a relation between the amount of money spent on advertising and profit.

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